

# Dynamical Mass Reduction in the Massive Yang-Mills Spectrum in $1 + 1$ dimensions

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The  $(1 + 1)$ -dimensional  $SU(N)$  Yang-Mills Lagrangian, with bare mass  $\mathcal{M}$ , and gauge coupling  $e$ , naively describes gluons of mass  $\mathcal{M}$ . In fact, renormalization forces  $\mathcal{M}$  to infinity. The system is in a confined phase, instead of a Higgs phase. The spectrum of this diverging-bare-mass theory contains particles of *finite* mass. There are an infinite number of physical particles, which are confined hadron-like bound states of fundamental colored excitations. These particles transform under irreducible representations of the global subgroup of the explicitly-broken gauge symmetry. The fundamental excitations are those of the  $SU(N) \times SU(N)$  principal chiral sigma model, with coupling  $g_0 = e/\mathcal{M}$ . We find the masses of meson-like bound states of two elementary excitations. This is done using the exact S matrix of the sigma model. We point out that the color-singlet spectrum coincides with that of the weakly-coupled anisotropic  $SU(N)$  gauge theory in  $2 + 1$  dimensions. We also briefly comment on how the spectrum behaves in the 't Hooft limit,  $N \rightarrow \infty$ .

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## I. INTRODUCTION

Yang-Mills theory in  $1 + 1$  dimensions has no local degrees of freedom. Introducing an explicit mass  $\mathcal{M}$  gives a theory of longitudinally-polarized gluons at tree level. It may seem intuitively obvious, for small gauge coupling, that a particle is either a vector Boson, with a mass roughly equal to  $\mathcal{M}$ , or a bound state of such vector Bosons. This intuition, however, is wrong. We show in this paper that the massive Yang-Mills theory describes an infinite number of particles, with masses that are much less than  $\mathcal{M}$ . This can be called dynamical mass reduction.

Alternatively, the massive Yang-Mills model can be thought of as a gauge field, coupled to an  $SU(N) \times SU(N)$  principal chiral nonlinear sigma model. The equivalence is seen by choosing the unitary gauge condition. In a perturbative treatment, the spin waves of the sigma model are Goldstone bosons, giving the vector particles a mass through the Higgs mechanism. Bardeen and Shizuya used this formulation in their proof of renormalizability [1].

The tree-level description fails because the excitations of the sigma model (without the gauge field) are not Goldstone Bosons. These excitations are massive. Introducing a gauge field produces a confining force between these excitations. There is no Higgs or Coulomb phase. There is only a confined phase.

We briefly describe some important earlier investigations of  $(1 + 1)$ -dimensional Yang-Mills theory. Non-Abelian gauge theories coupled to adjoint matter were studied with light-cone methods by Dalley and Klebanov [2]. This led to further investigations of gauged massive adjoint fermions [3]. Some detailed results for the spectrum of the model with of adjoint scalars were found later [4]. Conformal-field-theory methods have recently been applied to the model with adjoint Fermions [5]. Much has also been learned about pure Yang-Mills theory in  $1 + 1$  dimensions [6], and its connections with representation theory.

Our model differs from the Bosonic matter theory of Refs. [3], [4], in that the matter field has a non-trivial self-interaction. This means that there are two scales in our problem; the mass gap of the sigma model and the gauge coupling. This is why a nonrelativistic analysis, in which the former is assumed much larger than the latter, can work. A full-fledged relativistic analysis is harder, though we discuss this problem in the last section of this paper. We wish to stress that we are not studying a massive deformation of pure Yang-Mills theory [6] at all. In fact, the situation is exactly the opposite. The deformation is the Yang-Mills action, not the mass term.

A quantum field theory of an  $SU(N)$  gauge field, coupled minimally to an adjoint matter field, can have distinct Higgs and confinement phases [7], separated by a phase boundary, for space-time dimension greater than two. If this dimension is two, however, there is only the confined phase. In the confined phase, the excitations are bound states of the massive particles of the sigma model. These massive particles are color multiplets of degeneracy  $N^2$  [8].

The action of the massive  $SU(N)$  Yang-Mills field in  $1 + 1$  dimensions is

$$S = \int d^2x \left( -\frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \frac{e^2}{2g_0^2} \text{Tr} A_\mu A^\mu \right), \quad (\text{I.1})$$

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where  $A_\mu$  is Hermitian and  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]$  with  $\mu, \nu = 0, 1$  and indices are raised by  $\eta^{\mu\nu}$ , where  $\eta^{00} = -\eta^{11} = 1$ ,  $\eta^{01} = \eta^{10} = 0$ . If we drop the cubic and quartic terms from (I.1), the particles are gluons with mass  $\mathcal{M} = e/g_0$ .

Let's now consider a closely-related field theory, namely the ungauged principal chiral sigma model, with action

$$S_{\text{PCSM}} = \int d^2x \frac{1}{2g_0^2} \text{Tr} \partial_\mu U^\dagger(x) \partial^\mu U(x), \quad (\text{I.2})$$

where the field  $U(x)$  is in the fundamental representation of  $\text{SU}(N)$ . The action (I.2) has a global  $\text{SU}(N) \times \text{SU}(N)$  symmetry, given by the transformation  $U(x) \rightarrow V_L U(x) V_R$ , where  $V_{L,R} \in \text{SU}(N)$ . This model is asymptotically free, and has a mass gap, which we call  $m$ . It is possible that this mass gap is generated by non-real saddle points of the functional integral [10]. The running bare coupling  $g_0$  is driven to zero, as the ultraviolet cut-off is removed.

We promote the left-handed  $\text{SU}(N)$  global symmetry of the sigma model to a local symmetry, by introducing the covariant derivative  $D_\mu = \partial_\mu - ieA_\mu$ , where  $A_\mu$  is a new Hermitian vector field that transforms as  $A_\mu \rightarrow V_L^\dagger(x) A_\mu V_L(x) - \frac{i}{e} V_L^\dagger(x) \partial_\mu V_L(x)$ . We do not gauge the right-handed symmetry. The action is now

$$S = \int d^2x \left[ -\frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2g_0^2} \text{Tr} (D_\mu U)^\dagger D^\mu U \right]. \quad (\text{I.3})$$

In the unitary gauge, with  $U(x) = 1$  everywhere, this action (I.3) reduces to (I.1). In the remainder of this paper, however, we will study (I.3) in the axial gauge.

In our opinion, it is best to think of the left-handed symmetry as (confined) color- $\text{SU}(N)$  and the right-handed symmetry as flavor- $\text{SU}(N)$ . Confinement of left-handed color means that only singlets of the left-handed color group exist in the spectrum. There are ‘‘mesonic’’ bound states, as well as ‘‘baryonic’’ bound states. The mesonic bound states have one elementary particle of the sigma model and one elementary antiparticle. The simplest baryonic bound states consist of  $N$  of these elementary particles, with no antiparticles. There are also more complicated bound states, which exist because there are excitations in the sigma model (with no gauge field) transforming as higher representations of the color group [8]. In this paper, we only discuss the mesonic states in detail.

Recently Gongyo and Zwanziger have studied the nearest-neighbor lattice version of the action (I.3) using Monte-Carlo simulations [9]. They computed the static potential (through the Wilson loop) at different values of the coupling. They find clear evidence of confinement and string breaking at small values of  $g_0^{-2}$  (this is proportional to the parameter  $\gamma$ , in their notation), but a nearly-flat potential at large values, closer to the continuum limit. They suggest their results may indicate a phase transition to a Higgs phase (although they do not assert that this is the case). We believe the explanation is the essential singularity of the mass gap as a function of the bare coupling. This mass, in an asymptotically-free theory, vanishes faster than any power of  $g_0$  as  $g_0 \rightarrow 0$ . Thus, string breaking occurs so readily, that it may be difficult to distinguish the two phases. In this paper, the distinction is clear, because we take very small gauge coupling, suppressing (though not eliminating) string breaking. The continuum gauge coupling  $e$  (with dimensions of mass) is assumed to be much smaller than the mass gap of the sigma model. There should be no phase transition as the gauge coupling is increased. We therefore expect that, for any gauge coupling and any value of  $g_0$ , there is only the confined phase. Gongyo and Zwanziger also computed the vector-Boson propagator (the two-point function of a composite field), and the order parameter  $U$  (in a particular gauge) and the susceptibility of the latter. The lightest bound-state masses could be found in the behavior of the vector-Boson propagator. This would make for an interesting comparison with our results.

A mesonic bound state, in the axial gauge, is a sigma-model particle-antiparticle pair, confined by a linear potential. The string tension is

$$\sigma = e^2 C_N, \quad (\text{I.4})$$

where  $C_N$  is the smallest eigenvalue of the Casimir operator of  $\text{SU}(N)$ . The mass gap is

$$M = 2m + E_0 \ll \mathcal{M},$$

where  $E_0$  is the smallest (positive) binding energy, and  $m$  is the mass of a sigma-model elementary excitation. This mass  $M$  is finite, for fixed  $m$ , as the ultraviolet cut-off is removed. In contrast, the bare Yang-Mills mass  $\mathcal{M}$ , which is proportional to  $1/g_0$ , diverges.

Our approach is similar to that of Ref. [11]. We find the wave function of an unbound particle-antiparticle pair, taking into account scattering at the origin. Next, we generalize this to the wave function of the pair, confined by a linear potential. The method is inspired by the determination of the spectrum of the two-dimensional Ising model in an external magnetic field [13]. More sophisticated approaches to this and other two-dimensional models of confinement [14], [15], [16], including fine structure (form factors) of the fundamental excitations, have been developed. We do not take into account decays or corrections to the spectrum from matrix elements with more fundamental excitations [17] in this paper. For a general review, see Ref. [18].

We briefly introduce the axial gauge formulation in the next section. In Section III we discuss the S-matrix of the principal chiral nonlinear sigma model, and find the free particle-antiparticle wave function, for color group  $SU(N)$ , for  $N > 2$ . In Section IV, we find the wave functions and bound-state spectrum of a confined pair, for  $N > 2$  (including  $N \rightarrow \infty$  [19]). We note that the results generalize the result of Ref. [11], on the spectrum of 2 + 1-dimensional anisotropic  $SU(2)$  gauge theories, to  $SU(N)$ . We treat the  $N = 2$  case separately in Section V. We present some conclusions and proposals for further work in the last section.

## II. THE AXIAL GAUGE FORMULATION AND THE CONFINED PHASE

Care is necessary to understand why the bare mass is not the physical mass. If the axial gauge  $A_1 = 0$ , is chosen, the action (I.3) is

$$S = \int d^2x \left[ \frac{1}{2} \text{Tr} (\partial_1 A_0)^2 + \frac{1}{2g_0^2} \text{Tr} (\partial_0 U^\dagger + ieU^\dagger A_0)(\partial_0 U - ieA_0 U) - \frac{1}{2g_0^2} \text{Tr} \partial_1 U^\dagger \partial_1 U \right].$$

Let us introduce the traceless Hermitian generators  $t_a$  of  $SU(N)$ ,  $a = 1, \dots, N^2 - 1$ , with normalization  $\text{Tr} t_a t_b = \delta_{ab}$  and structure coefficients  $f_{abc}$ , defined by  $[t_b, t_c] = if_{abc} t_a$ . If we naively eliminate  $A_0$ , by its equation of motion (or integrate  $A_0$  from the functional integral), we obtain the effective action

$$S = \int d^2x \left( \frac{1}{2g_0^2} \text{Tr} \partial_\mu U^\dagger \partial^\mu U + \frac{1}{2} j_0^L \frac{1}{-\partial_1^2 + e^2/g_0^2} j_0^L \right), \quad (\text{II.1})$$

where  $j_\mu^L(x)_b = -i \text{Tr} t_b \partial_\mu U(x) U^\dagger(x)$  is the Noether current of the left-handed  $SU(N)$  symmetry. The potential induced on the color-charge density, in the second term of (II.1), indicates that charges are screened, instead of confined. This conclusion, however, is based on the fact that  $U^\dagger U = 1$ . In the renormalized theory,  $U$  is not a physical field. The physical scaling field of the principal chiral nonlinear sigma model is not a unitary matrix. This fact is discussed more explicitly in Refs. [20], in the limit  $N \rightarrow \infty$ , with  $g_0^2 N$  fixed. The actual excitations of the principal chiral model are massive, with a left and right color charge [8], so that no screening takes place.

A more careful approach is to first find the Hamiltonian in the temporal gauge  $A_0 = 0$ . Gauge invariance, or Gauss' law, must be imposed on physical states. The Hamiltonian is

$$H = \int dx^1 \left\{ \frac{g_0^2}{2} [j_0^L(x^1)_b]^2 + \frac{1}{2g_0^2} [j_1^L(x^1)_b]^2 + \frac{1}{2} [E(x^1)_b]^2 + \frac{e}{g_0^2} j_1^L(x^1)_b A_1(x^1)_b \right\}, \quad (\text{II.2})$$

where  $A_1(x^1)_b = \text{Tr} t_b A$  and  $E_a$  is the electric field, obeying  $[E(x^1)_a, A_1(y^1)_b] = -i\delta_{ab}\delta(x^1 - y^1)$ . The Hamiltonian (II.2) must be supplemented by Gauss' law  $G(x^1)_a \Psi = 0$ , for any physical state  $\Psi$ , where  $G(x^1)_a$  is the generator of spatial gauge transformations:

$$G(x^1)_a = \partial_1 E(x^1)_a + ef_{abc} A_1(x^1)^b E(x^1)_c - \frac{e}{g_0^2} j_0^L(x^1)_a. \quad (\text{II.3})$$

If we require that the electric field vanishes at the boundaries  $x^1 = \pm l/2$ , Gauss' law may be explicitly solved [12], to yield the expression for the electric field:

$$E(x^1)_a = \int_{-l/2}^{x^1} dy^1 \left\{ \mathcal{P} \exp \left[ ie \int_{-l/2}^{y^1} dz^1 \mathcal{A}_1(z^1) \right] \right\}_a^b \frac{e}{g_0^2} j_0^L(y^1)_b, \quad (\text{II.4})$$

where  $\mathcal{A}_1(x^1)_a^b = if_{abc} A_1(x^1)_c$  is the gauge field in the adjoint representation. There remains a global gauge invariance, which must be satisfied by physical states, *i.e.*,  $\Gamma_a \Psi = 0$ , where

$$\Gamma_a = \int_{-l/2}^{l/2} dy^1 \left\{ \mathcal{P} \exp \left[ ie \int_{-l/2}^{y^1} dz^1 \mathcal{A}_1(z^1) \right] \right\}_a^b \frac{e}{g_0^2} j_0^L(y^1)_b. \quad (\text{II.5})$$

Now we are free to chose  $A_1(x^1)_b = 0$ , which simplifies (II.4) and (II.5). The solution for the electric field yields the Hamiltonian

$$H = \int dx^1 \left\{ \frac{g_0^2}{2} [j_0^L(x^1)_b]^2 + \frac{1}{2g_0^2} [j_1^L(x^1)_b]^2 \right\} - \frac{e^2}{2g_0^4} \int dx^1 \int dy^1 |x^1 - y^1| j_0^L(x^1)_b j_0^L(y^1)_b, \quad (\text{II.6})$$

where in the last step, we have taken the size  $l$  of the system to infinity. The last term is a linear potential which confines left-handed color. Notice that (II.6) is not bounded from below on the full Hilbert space. This is because of the last, nonlocal term; the energy can be lowered by adding pairs of colored particles (or antiparticles) and by separating them. The residual Gauss-law condition  $\Gamma_a \Psi = 0$ , forces the global left-handed color to be a singlet, thereby removing the instability,

### III. THE FREE PARTICLE-ANTIPARTICLE WAVE FUNCTION: $N > 2$

The quantized principal chiral nonlinear sigma model is integrable. This property, together with physical considerations, has been used to find the exact S-matrix [8].

An excitation has rapidity  $\theta$ , related to that excitation's energy and momentum, by  $E = m \sinh \theta$  and  $p = m \cosh \theta$ , respectively.

Let us consider a state with two excitations. One excitation is an antiparticle of rapidity  $\theta_1$  and left and right  $SU(N)$  color indices  $a_1, b_1 = 1, \dots, N$ , respectively. The second excitation is a particle of rapidity  $\theta_2$ , and left and right color indices  $a_2, b_2$ , respectively. Explicitly the state is

$$|A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2\rangle_{\text{in}}.$$

The S-matrix element,  $S(\theta)_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1}$ , is defined by

$${}_{\text{out}}\langle A, \theta'_1, d_1, c_1; P, \theta'_2, c_2, d_2 | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2 \rangle_{\text{in}} = S(\theta)_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} 4\pi\delta(\theta_1 - \theta'_1) 4\pi\delta(\theta_2 - \theta'_2),$$

where  $\theta = \theta_1 - \theta_2$ . This S-matrix element is [8]

$$S(\theta)_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} = S(\theta) \left[ \delta_{a_1}^{c_1} \delta_{a_2}^{c_2} - \frac{2\pi i}{N(\pi i - \theta)} \delta_{a_1 a_2} \delta^{c_1 c_2} \right] \left[ \delta_{b_1}^{d_1} \delta_{b_2}^{d_2} - \frac{2\pi i}{N(\pi i - \theta)} \delta_{b_1 b_2} b^{d_1 d_2} \right],$$

where

$$S(\theta) = \frac{\sinh \left[ \frac{(\pi i - \theta)}{2} - \frac{\pi i}{N} \right]}{\sinh \left[ \frac{(\pi i - \theta)}{2} + \frac{\pi i}{N} \right]} \left\{ \frac{\Gamma[i(\pi i - \theta)/2\pi + 1] \Gamma[-i(\pi i - \theta)/2\pi - 1/N]}{\Gamma[i(\pi i - \theta)/2\pi + 1 - 1/N] \Gamma[-i(\pi i - \theta)/2\pi]} \right\}^2. \quad (\text{III.1})$$

For  $N > 2$ , the expression (III.1) may be written in the exponential form [23] :

$$S(\theta) = \exp 2 \int_0^\infty \frac{d\xi}{\xi \sinh \xi} \left[ 2(e^{2\xi/N} - 1) - \sinh(2\xi/N) \right] \sinh \frac{\xi\theta}{\pi i}. \quad (\text{III.2})$$

We will discuss the  $N = 2$  case separately in Section V.

The wave function of a free antiparticle at  $x^1$  and a free particle at  $x^2$ , with momenta  $p_1$  and  $p_2$ , respectively, is

$$\Psi_{p_1, p_2}(x^1, y^1)_{a_1 a_2; b_1 b_2} = \begin{cases} e^{ip_1 x^1 + ip_2 y^1} A_{a_1 a_2; b_1 b_2}, & \text{for } x^1 < y^1, \\ e^{ip_2 x^1 + ip_1 y^1} S(\theta)_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} A_{c_1 c_2; d_1 d_2}, & \text{for } x^1 > y^1. \end{cases} \quad (\text{III.3})$$

where  $A_{a_1 a_2; b_1 b_2}$  is set of arbitrary complex numbers.

The residual Gauss' law in the axial gauge,  $\Gamma_a \Psi = 0$ , restricts physical states to those which are invariant under global left-handed  $SU(N)$  color transformations. This means that the particle-antiparticle state of the form (III.3) must be projected to a global left-color singlet. A left-color-singlet wave function is

$$\Psi_{p_1 p_2}(x^1, y^1)_{b_1 b_2} = \delta^{a_1 a_2} \Psi_{p_1, p_2}(x^1, y^1)_{a_1 a_2; b_1 b_2}. \quad (\text{III.4})$$

There are states of degeneracy  $N^2 - 1$ , which resemble massive gluons. These transform as the adjoint representation of the right-handed color symmetry. The wave function of such a state is traceless in the right-handed color indices:

$$\delta^{b_1 b_2} \Psi_{p_1 p_2}(x^1, y^1)_{b_1 b_2} = 0. \quad (\text{III.5})$$

We use a non-relativistic approximation  $p_{1,2} \ll m$ . The wave function in this limit becomes

$$\Psi_{p_1 p_2}(x^1, y^1)_{b_1 b_2} = \begin{cases} e^{ip_1 x^1 + ip_2 y^1} A_{b_1 b_2}, & \text{for } x^1 < y^1, \\ e^{ip_2 x^1 + ip_1 y^1} \exp(i\pi - \frac{ih_N}{\pi m} |p_1 - p_2|) A_{b_1 b_2}, & \text{for } x^1 > y^1. \end{cases} \quad (\text{III.6})$$

where  $\text{Tr} A = 0$ , and

$$\begin{aligned} h_N &= 2 \int_0^\infty \frac{d\xi}{\sinh \xi} \left[ 2(e^{2\xi/N} - 1) - \sinh(2\xi/N) \right] \\ &= -4\gamma - \psi\left(\frac{1}{2} + \frac{1}{N}\right) - 3\psi\left(\frac{1}{2} - \frac{1}{N}\right) - 4 \ln 4, \end{aligned} \quad (\text{III.7})$$

where  $\gamma$  is the Euler-Mascheroni constant, and  $\psi(x) = d \ln \Gamma(x)/dx$  is the digamma function. The expression in (III.6) must be equal to the wave function of two confined particles for sufficiently small  $|x^1 - y^1|$ . To compare the two expressions, it is convenient to use center-of-mass coordinates,  $X$ ,  $x$ , and their respective momenta  $P$ ,  $p$ . Explicitly,  $X = x^1 + y^1$ ,  $x = y^1 - x^1$ ,  $P = p_1 + p_2$  and  $p = p_2 - p_1$ . In these coordinates, the wave function is

$$\Psi_p(x)_{b_1 b_2} = \begin{cases} \cos(px + \omega) A_{b_1 b_2}, & \text{for } x > 0, \\ \cos[-px + \omega - \phi(p)] A_{b_1 b_2}, & \text{for } x < 0, \end{cases} \quad (\text{III.8})$$

for some constant  $\omega$ , with the phase shift  $\phi(p) = \pi - \frac{h_N}{\pi m} |p|$ .

Another type of mesonic state is the right-handed color singlet, with  $A_{b_1 b_2} = \delta_{b_1 b_2}$ . The non-relativistic limit of the wave function in this case is

$$\Psi_p(x)_{\text{singlet}} = \begin{cases} \cos(px + \omega), & \text{for } x > 0, \\ \cos[-px + \omega - \chi(p)], & \text{for } x < 0, \end{cases} \quad (\text{III.9})$$

where  $\chi(p) = -\frac{h_N}{\pi m} |p|$ .

#### IV. MESONIC STATES OF MASSIVE YANG-MILLS THEORY: $N > 2$

The wave function of a particle-antiparticle pair, confined by string tension  $\sigma$ , satisfies the Schroedinger equation

$$-\frac{1}{m} \frac{d^2}{dx^2} \Psi(x)_{b_1 b_2} + \sigma |x| \Psi(x)_{b_1 b_2} = E \Psi(x)_{b_1 b_2}, \quad (\text{IV.1})$$

where  $E$  is the binding energy [13]. The solution to Equation (IV.1) is

$$\Psi(x)_{b_1 b_2} = \begin{cases} C \text{Ai} \left[ (m\sigma)^{\frac{1}{3}} \left( x + \frac{E}{\sigma} \right) \right] A_{b_1 b_2}, & \text{for } x > 0 \\ C' \text{Ai} \left[ (m\sigma)^{\frac{1}{3}} \left( -x + \frac{E}{\sigma} \right) \right] A_{b_1 b_2}, & \text{for } x < 0, \end{cases} \quad (\text{IV.2})$$

where  $\text{Ai}(x)$  is the Airy function of the first kind, and  $C$ ,  $C'$  are constants.

For  $|x| \ll (m\sigma)^{-1/3}$ , the potential energy in (IV.1) is sufficiently small that the wave function is (III.8), with  $|p| = (mE)^{\frac{1}{2}}$ . The wave function (IV.2) is approximated in this region by

$$\Psi(x)_{b_1 b_2} = \begin{cases} C \frac{1}{(x + \frac{E}{\sigma})^{\frac{1}{4}}} \cos \left[ \frac{2}{3} (m\sigma)^{\frac{1}{2}} \left( x + \frac{E}{\sigma} \right)^{\frac{3}{2}} - \frac{\pi}{4} \right] A_{b_1 b_2}, & \text{for } x > 0, \\ C' \frac{1}{(-x + \frac{E}{\sigma})^{\frac{1}{4}}} \cos \left[ -\frac{2}{3} (m\sigma)^{\frac{1}{2}} \left( -x + \frac{E}{\sigma} \right)^{\frac{3}{2}} + \frac{\pi}{4} \right] A_{b_1 b_2}, & \text{for } x < 0. \end{cases}$$

Let us now consider the  $(N^2 - 1)$ -plet of mesonic states. The wave functions (III.8) and (IV.2) should be the same for  $x \downarrow 0$ , yielding

$$\frac{C}{(\frac{E}{\sigma})^{\frac{1}{4}}} \cos \left[ \frac{2}{3} (m\sigma)^{\frac{1}{2}} \left( \frac{E}{\sigma} \right)^{\frac{3}{2}} - \frac{\pi}{4} \right] = \cos(\omega). \quad (\text{IV.3})$$

Equation (IV.3) implies

$$C = \left( \frac{E}{\sigma} \right)^{\frac{1}{4}}, \quad \omega = \frac{2}{3} (m\sigma)^{\frac{1}{2}} \left( \frac{E}{\sigma} \right)^{\frac{3}{2}} - \frac{\pi}{4}.$$

The wave functions (III.8) and (IV.2) should also be the same for  $x \uparrow 0$ , yielding

$$\frac{C'}{(\frac{E}{\sigma})^{\frac{1}{4}}} \cos \left[ -\frac{2}{3} (m\sigma)^{\frac{1}{2}} \left( \frac{E}{\sigma} \right)^{\frac{3}{2}} + \frac{\pi}{4} \right] = \cos \left[ \omega - \pi + \frac{h_N}{\pi m} (mE)^{\frac{1}{2}} \right], \quad (\text{IV.4})$$

hence  $C' = C = \left(\frac{E}{\sigma}\right)^{\frac{1}{4}}$ . The arguments of the cosine on each side of (IV.4) must be the same, modulo  $2\pi$ :

$$-\frac{2}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}} + \frac{\pi}{4} + 2\pi n = \frac{2}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}} - \frac{5\pi}{4} + \frac{h_N}{\pi m}(mE)^{\frac{1}{2}},$$

for  $n = 0, 1, 2, \dots$ . We simplify this to

$$\frac{4}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}} + \frac{h_N}{\pi m}(mE)^{\frac{1}{2}} - \left(n + \frac{3}{4}\right)2\pi = 0. \quad (\text{IV.5})$$

An analysis which is similar to that of the previous paragraph yields the quantization condition for the right-handed singlet state (III.9). This is

$$\frac{4}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}} + \frac{h_N}{\pi m}(mE)^{\frac{1}{2}} - \left(n + \frac{1}{4}\right)2\pi = 0. \quad (\text{IV.6})$$

Equations (IV.5) and (IV.6) are depressed cubic equations of the variable  $Z_n = E_n^{\frac{1}{2}}$ . These cubic equations have only one real solution for each value of  $n$ , because  $h_N/(\pi m^{\frac{1}{2}}) > 0$ . The solution of Equations (IV.5) and (IV.6) is

$$E_n = \left\{ \left[ \epsilon_n + (\epsilon_n^2 + \beta_N^3)^{\frac{1}{2}} \right]^{\frac{1}{3}} + \left[ \epsilon_n - (\epsilon_n^2 + \beta_N^3)^{\frac{1}{2}} \right]^{\frac{1}{3}} \right\}^{\frac{1}{2}}, \quad (\text{IV.7})$$

where

$$\epsilon_n = \frac{3\pi}{4} \left(\frac{\sigma}{m}\right)^{\frac{1}{2}} \left(n + \frac{1}{2} \pm \frac{1}{4}\right), \quad \beta_N = \frac{h_N \sigma^{\frac{1}{2}}}{4\pi m}, \quad (\text{IV.8})$$

where  $\pm = +$  for the  $(N^2 - 1)$ -plet, and  $\pm = -$  for the singlet.

We show in the next section that the expressions (IV.7) and (IV.8) remain valid for the  $SU(2)$  case, with  $h_2 = -4 \ln 2 + 2$  and, significantly, with a reversal of the sign in (IV.8). For  $N = 2$  *only* we must take  $\pm = -$  for the  $(N^2 - 1)$ -plet (the triplet) and  $\pm = +$  for the singlet.

As it happens, the results we have just obtained for the singlet spectrum generalize the result of Ref. [11], on the spectrum of  $2 + 1$ -dimensional anisotropic  $SU(2)$  gauge theories, to  $SU(N)$  (where  $\sigma$  is replaced by  $2\sigma$ ).

Another interesting special case is the 't Hooft limit  $N \rightarrow \infty$  [20], [24]. The mass gap of the sigma model should be fixed in this limit. The string tension  $\sigma$  will be fixed as well [19], provided  $e^2 N$  is fixed. In this limit  $h_N \rightarrow 0$ , and we find

$$E_n = \left[ \frac{3\pi}{2} \left(\frac{\sigma}{m}\right)^{\frac{1}{2}} \left(n + \frac{1}{2} \pm \frac{1}{4}\right) \right]^{1/3}. \quad (\text{IV.9})$$

## V. THE $N = 2$ CASE

The exponential expression for the S-matrix (III.2) is only correct for  $N > 2$ . The principal chiral model with  $SU(2) \times SU(2)$  symmetry is equivalent to the  $O(4)$ -symmetric nonlinear sigma model. We will express the S matrix, first found in Ref. [21], by an exponential expression [22].

A state with one excitation has a left-handed color index  $a = 1, 2$  and a right-handed color index  $b = 1, 2$ . In the  $O(4)$  formulation, excitations have a single species index  $j = 1, 2, 3, 4$ . The  $SU(2) \times SU(2)$ -symmetric states are related to the  $O(4)$ -symmetric states by

$$\begin{aligned} |P, \theta, a, b\rangle_{\text{in}} &= \sum_j \frac{1}{\sqrt{2}} \left( \delta^{j4} \delta_{ab} - i\sigma_{ab}^j \right) |\theta, j\rangle_{\text{in}}, \\ |A, \theta, a, b\rangle_{\text{in}} &= \sum_j \frac{1}{\sqrt{2}} \left( \delta^{j4} \delta_{ab} - i\sigma_{ab}^j \right)^* |\theta, j\rangle_{\text{in}}, \end{aligned}$$

where  $\sigma^j$  with  $j = 1, 2, 3$  are the Pauli matrices. The  $O(4)$  two-excitation S-matrix,  $S(\theta)_{j_1' j_2'}$  is given by

$${}_{\text{out}} \langle \theta_1', j_1'; \theta_2', j_2' | \theta_1, j_1; \theta_2, j_2 \rangle_{\text{in}} = S(\theta)_{j_1' j_2'}^{j_1 j_2} 4\pi \delta(\theta_1 - \theta_1') 4\pi \delta(\theta_2 - \theta_2'),$$

where [22]

$$S(\theta)_{j'_1 j'_2}^{j_1 j_2} = \left[ \frac{\theta + \pi i}{\theta - \pi i} (P^0)_{j'_1 j'_2}^{j_1 j_2} + \frac{\theta - \pi i}{\theta + \pi i} (P^+)_{j'_1 j'_2}^{j_1 j_2} + (P^-)_{j'_1 j'_2}^{j_1 j_2} \right] Q(\theta),$$

$$Q(\theta) = \exp 2 \int_0^\infty \frac{d\xi}{\xi} \frac{e^{-\xi} - 1}{e^\xi + 1} \sinh \left( \frac{\xi \theta}{\pi i} \right),$$

and  $P^0$ ,  $P^+$ , and  $P^-$  are the singlet, symmetric-traceless, and antisymmetric projectors, which are

$$(P^0)_{j'_1 j'_2}^{j_1 j_2} = \frac{1}{4} \delta^{j_1 j_2} \delta_{j'_1 j'_2}, \quad (P^+)_{j'_1 j'_2}^{j_1 j_2} = \frac{1}{2} (\delta_{j'_1}^{j_1} \delta_{j'_2}^{j_2} + \delta_{j'_2}^{j_1} \delta_{j'_1}^{j_2}) - \frac{1}{4} \delta^{j_1 j_2} \delta_{j'_1 j'_2},$$

$$(P^-)_{j'_1 j'_2}^{j_1 j_2} = \frac{1}{2} (\delta_{j'_1}^{j_1} \delta_{j'_2}^{j_2} - \delta_{j'_2}^{j_1} \delta_{j'_1}^{j_2}),$$

respectively.

We write the left-color-singlet wave function for a free particle and antiparticle:

$$\Psi_{p_1, p_2}(x^1, y^1)_{b_1 b_2} = D_{b_1 b_2}^{j_1 j_2} \begin{cases} e^{ip_1 x^1 + ip_2 y^1} A_{j_1 j_2}, & \text{for } x^1 > y^1 \\ e^{ip_2 x^1 + ip_1 y^1} S(\theta)_{j'_1 j'_2}^{j_1 j_2} A_{j'_1 j'_2}, & \text{for } x^1 < y^1, \end{cases} \quad (\text{V.1})$$

where

$$D_{b_1 b_2}^{j_1 j_2} = \frac{1}{2} \delta^{a_1 a_2} \left( \delta^{j_1 4} \delta_{a_1 b_1} - i \sigma_{a_1 b_1}^{j_1} \right)^* \left( \delta^{j_2 4} \delta_{a_2 b_2} - i \sigma_{a_2 b_2}^{j_2} \right).$$

There is a triplet of degenerate states and one singlet state. The triplet satisfies

$$\delta^{b_1 b_2} \Psi_{p_1, p_2}(x^1, y^1)_{b_1 b_2} = 0. \quad (\text{V.2})$$

Substituting (V.1) into (V.2) gives the condition

$$\delta^{b_1 b_2} D_{b_1 b_2}^{j_1 j_2} A_{j_1 j_2} = \delta^{j_1 j_2} A_{j_1 j_2} = 0.$$

The traceless matrix  $A_{j_1 j_2}$  can be split into a symmetric and an antisymmetric part,  $A_{j_1 j_2}^+ = (A_{j_1 j_2} + A_{j_2 j_1})/2$  and  $A_{j_1 j_2}^- = (A_{j_1 j_2} - A_{j_2 j_1})/2$ , respectively. The matrix  $A_{j_1 j_2}^+$ , however, does not contribute to the wave function (V.1), because

$$D_{b_1 b_2}^{j_1 j_2} A_{j_1 j_2}^+ = \frac{1}{2} \delta_{b_1 b_2} \text{Tr } A^+ = 0.$$

The matrix  $A_{j_1 j_2}^-$  satisfies [21], [22]:

$$S(\theta)_{j'_1 j'_2}^{j_1 j_2} A_{j_1 j_2}^- = Q(\theta) A_{j'_1 j'_2}^-. \quad (\text{V.3})$$

Substituting (V.3) into (V.1), in center-of-mass coordinates and the non-relativistic limit, we find

$$\Psi_p(x)_{b_1 b_2} = D_{b_1 b_2}^{j_1 j_2} \begin{cases} \cos(px + \omega) A_{j_1 j_2}, & \text{for } x > 0, \\ \cos[-px + \omega - \phi(p)] A_{j_1 j_2}, & \text{for } x < 0, \end{cases} \quad (\text{V.4})$$

where  $\phi(p) = -\frac{i h_2}{\pi m} |p|$ , where

$$h_2 = 2 \int_0^\infty d\xi \frac{e^{-\xi} - 1}{e^\xi + 1} = -4 \ln 2 + 2. \quad (\text{V.5})$$

The wave function of the right-color-singlet bound state is

$$\Psi_{p_1, p_2}^{\text{singlet}}(x^1, y^1) = \begin{cases} e^{ip_1 x^1 + ip_2 y^1}, & \text{for } x^1 > y^1, \\ e^{ip_2 x^1 + ip_1 y^1} \frac{\theta + \pi i}{\theta - \pi i} Q(\theta), & \text{for } x^1 < y^1. \end{cases} \quad (\text{V.6})$$

In center-of-mass coordinates, in the non-relativistic approximation, this becomes

$$\Psi_p^{\text{singlet}}(x) = \begin{cases} \cos(px + \omega), & \text{for } x > 0, \\ \cos[-px + \omega - \chi(p)], & \text{for } x < 0, \end{cases} \quad (\text{V.7})$$

where  $\chi(p) = \pi - \frac{ih_2}{\pi m}|p|$ .

From this point onward, the analysis is similar to what we've presented in the last two sections. We obtain (IV.7), (IV.8), except that  $h_N$  (defined in (III.7)) is replaced with  $h_2$  (defined in (V.5)), with one important difference; we have  $\pm = +$  for the singlet and  $\pm = -$  for the triplet in Eq. (IV.8). As mentioned at the end of the last section, the singlet spectrum coincides with that of Ref. [11], in which  $\sigma$  must be replaced by  $2\sigma$ .

## VI. CONCLUSIONS AND OUTLOOK

We have found the spectrum of massive  $(1 + 1)$ -dimensional  $SU(N)$  Yang-Mills theory, for small gauge coupling. To do this, we formulated the model as a principal chiral sigma model coupled to a massless Yang-Mills field. In the axial gauge, there are sigma-model particles and antiparticles which bind to make left-color singlets. We obtained the mesonic spectrum by determining the particle-antiparticle wave function in the non-relativistic limit, taking into account the phase shift at the origin.

In the future, we would like to find relativistic corrections to the mass spectrum. This was done in Ref. [16] for the Ising model in an external magnetic field. The goal would be to find mesonic eigenstates of the Hamiltonian (II.6) of the form:

$$|\Psi_B\rangle_{b_1 b_2} = |\Psi_B^{(2)}\rangle_{b_1 b_2} + |\Psi_B^{(4)}\rangle_{b_1 b_2} + |\Psi_B^{(6)}\rangle_{b_1 b_2} + \dots,$$

where the state  $|\Psi_B^{(2M)}\rangle_{b_1 b_2}$  contains  $M$  particles and  $M$  antiparticles. The multi-particle contributions are included because an electric string may break [17], producing pairs of sigma-model excitations. Nonetheless, for small gauge coupling, the “two-quark” approximation is valid. In the this approximation, the bound state is treated as

$$\begin{aligned} |\Psi_B\rangle_{b_1 b_2} &\approx |\Psi_B^{(2)}\rangle_{b_1 b_2} = \frac{1}{2} \int \frac{d\theta_1}{4\pi} \frac{d\theta_2}{4\pi} \Psi(p_1, p_2)_{a_2 a_2} |A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2\rangle, \text{ where,} \\ \Psi(p_1, p_2)_{a_1 a_2} &= S(\theta) \left[ \delta_{a_1}^{c_1} \delta_{a_2}^{c_2} - \frac{2\pi i}{N(\pi i - \theta)} \delta_{a_1 a_2} \delta^{c_1 c_2} \right] \Psi(p_2, p_1)_{c_1 c_2}. \end{aligned} \quad (\text{VI.1})$$

The spectrum of masses  $\Delta$ , of the states (VI.1) is found from the Bethe-Salpeter equation  $(H - \Delta)|\Psi_B^{(2)}\rangle_{b_1 b_2} = 0$ . Acting on this state with the Hamiltonian (II.6) yields

$$\begin{aligned} (m \cosh \theta_1 + m \cosh \theta_2 - \Delta) \Psi(p'_1, p'_2)_{c_1 c_2} \delta_{b_1 d_1} \delta_{b_2 d_2} \\ = \frac{e^2}{4g_0^4} \int \frac{d\theta_1}{4\pi} \frac{d\theta_2}{4\pi} \Psi(p_1, p_2)_{a_1 a_2} \int dx^1 dy^1 |x^1 - y^1| \\ \times \langle A, \theta'_1, d_1, c_1; P, \theta'_2, c_2, d_2 | \text{Tr} [j_0^L(x^1) j_0^L(y^1)] | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2 \rangle, \end{aligned} \quad (\text{VI.2})$$

where the operator  $\text{Tr} [j_0^L(x^1) j_0^L(y^1)]$  is not time-ordered. The matrix element

$$\langle A, \theta'_1, d_1, c_1; P, \theta'_2, c_2, d_2 | \text{Tr} [j_0^L(x^1) j_0^L(y^1)] | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2 \rangle$$

is obtained by inserting a complete set of states between the current operators and using the exact form factors of the currents of the principal chiral sigma model. For finite  $N$ , only the leading two-particle form factors of currents are known [23] and only a vacuum insertion can be made. The complete matrix element is known at large  $N$  [24], which should help in finding the relativistic corrections to the eigenvalues of Eq. (VI.2).

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